Int. J. Solids Structures, 1970, Vol. 6, pp. 587 to 604. Pergamon Press. Printed in Great Britain

BUCKLING, POST-BUCKLING AND LIMIT ANALYSIS OF COMPLETELY SYMMETRIC ELASTIC STRUCTURES†

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Abstract—The buckling mode of a structure is defined to be symmetric if its sign is indefinite; this happens when the potential energy expansion near the buckling point does not contain terms which are cubic in the buckling mode. If, in addition, the cubic terms vanish identically for all possible modes then the structure is defined to be "completely symmetric". Many structures of technical significance are included in this definition, such as columns, plates, frameworks, etc.

If certain technically realistic order-of-magnitude assumptions are made, the analysis of the buckling and post-buckling behavior of completely symmetric structures can be carried out in great generality. For example, it is shown in the present paper that structures of this type buckle under increasing loads and are therefore insensitive to initial imperfections. The post-buckling state is characterized by the satisfaction of a minimum complementary energy principle, which represents an extension of the corresponding classical principle into the nonlinear domain. Moreover, the energy can be bracketed between upper and lower bounds and an error estimate is thus established at least in an averaging sense.

Under certain circumstances the load approaches a finite value as the structure approaches collapse. This collapse load can also be bracketed between classes of "statically admissible" load parameters (representing lower bounds) and "kinematically admissible" load parameters (representing upper bounds). The gap between these bounds can be reduced arbitrarily.

The example of a slender statically indeterminate beam subjected to lateral and torsional buckling is introduced to demonstrate the general principles developed in the paper.

1. INTRODUCTION

THE "CLASSICAL" theory of the stability of elastic structures was substantially completed in the first part of the current century [1]. A major advance, although barely noticed at the time because of the German occupation of the Netherlands, occurred with the publication of Koiter's doctoral thesis [2] in 1945. This work, itself now a classic, has served as focal point for a large number of further investigations [e.g. 3–11] of the stability of the critical buckling point itself or, equivalently, of the behavior of the structure immediately after buckling. The problem is now well understood, at least in principle.

In contrast, few general conclusions appear to have been derived thus far about the behavior of structures with large buckling deformations. The present study concerns itself with this problem. It will be shown that the post-buckling response of certain types of structures is governed by boundedness theorems which make it possible to develop approximate analytical techniques with error estimates. Moreover, under certain circumstances these structures may collapse as the loads approach limiting values. Although elastic behaviour is postulated, even in the limit, these collapse loads can be bracketed

† This work was supported by NSF Grant GK-368 to the University of Illinois at Chicago Circle, Chicago, Illinois.

between upper and lower bounds in a manner which is reminiscent of the methods applicable to the theory of perfect plasticity.

In essence, Koiter's contribution consisted in developing a perturbation expansion of the potential energy near the critical point of bifurcation of the equilibrium configuration in terms of an amplitude parameter. With the first-order terms automatically vanishing (as they do for all points of equilibrium), bifurcation occurs when the smallest admissible value (identified with the buckling mode) of the second-order term vanishes. The question of stability is then governed by the third and fourth order terms, although, under exceptional circumstances, even higher-order terms may become significant. In particular, if the third-order term in the expansion of the potential energy relative to the actual buckling mode does not vanish, the point of bifurcation is unstable. If it does vanish, stability is determined by the smallest admissible value of the fourth-order term. In this case a structure may be designated to be "symmetric" in the sense that two buckling configurations (identical except for their sign) are then equally possible.

2. BASIC DEVELOPMENTS

In the present paper we define a structure to be "completely symmetric" if the third-order term in the expansion of the potential energy vanishes identically, rather than in relation to only the actual buckling mode. Technically significant examples of completely symmetric structures include columns, plates, many types of frameworks and trusses, and beams buckling laterally; counter-examples are arches and shells, which may nevertheless be "symmetric".

The establishment of complete symmetry is often predicated on the validity and technical realism of certain order-of-magnitude assumptions. That is, we divide the displacements describing the configuration of a structure into "primary" ones (designated by u) which are small in the sense that they appear only linearly in the strain-displacement relations, and "buckling" displacements (designated by v) which require the inclusion of quadratic terms. This is a reasonable assumption and lies at the root of most technical buckling theories. A structure which buckles elastically always exhibits some geometric pathology in that some dimensions are much smaller than others, as is obviously the case with "slender" columns, "thin" plates, etc. It is therefore reasonable to assume that the displacement fields exhibit corresponding inequalities. For example, in the case of a buckled column the axial displacement u is much smaller than the lateral displacement v. If a beam buckles laterally under the effect of bending moments and/or axial thrust, the axial deformation and the deflection in the strong direction is primary and is therefore designated, collectively, by u; the collective designation v applies to the vector whose components are, respectively, the deflection in the weak direction and the rotation.

Associated with these two types of displacement fields are two types of generalized strain fields $\mathbf{e}$ and $\mathbf{k}$, respectively, so that the strain energy density (per unit length, area, etc.) can be written in the form

$$U = U_2(\mathbf{e}) + U'_2(\mathbf{k}) \tag{1}$$

in which both U_2 and U'_2 are assumed to be quadratic in their argument for the purposes of the present paper. For example, U_2 may represent the membrane energy and U'_2 the bending energy in a plate, with $\mathbf{e}$ and $\mathbf{k}$ designating the membrane strain and curvature tensors, respectively.

In line with the previous discussion the strain–displacement relations are of the form

$$\begin{aligned} \mathbf{e} &= \mathbf{l}_1(u)+\tfrac{1}{2}\mathbf{l}_2(v) \\ \mathbf{k} &= \mathbf{k}(v) \end{aligned} \tag{2}$$

in which $\mathbf{l}_1$ and $\mathbf{k}$ are linear and $\mathbf{l}_2$ quadratic in their arguments. In order that the structure may be "completely symmetric" it is essential that no linear term in v be present in the expression for $\mathbf{e}$; straight beams and plates, for example, conform to this requirement, but arches and shells do not.

The possibility of buckling in the sense of bifurcation may occur if the loads act only on the primary displacements u. Assuming, without loss of significant generality, that all kinematic constraints are workless, we then write the boundary conditions in the form

$$\begin{aligned} W &= \lambda W_1(u) \quad && \text{on } B_t \\ u &= 0 && \text{on } B_u \\ L(v) &= 0 && \text{on } B \end{aligned} \tag{3}$$

in which λ is a load parameter, L a system of linear operators, and in which $B = B_t + B_u$ represents the total boundary. W is the work done by the loads, and W_1 is linear in its argument.

Generalized stresses $\mathbf{t}$ and $\mathbf{m}$ (denoting, for example, membrane force and bending moment tensors, respectively) are defined by means of the linear constitutive relations

$$\begin{aligned} \mathbf{t} &= \frac{\mathrm{d}U_2}{\mathrm{d}\mathbf{e}} \\ \mathbf{m} &= \frac{\mathrm{d}U_2'}{\mathrm{d}\mathbf{k}}. \end{aligned} \tag{4}$$

We note that the internal work density is given by the sum of the inner products $\mathbf{t}\,.\,\mathbf{e}$ and $\mathbf{m}\,.\,\mathbf{k}$.

In the absence of body forces the potential energy†

$$\Omega = U - W \tag{5}$$

serves to express the equations of equilibrium in the variational form

$$\delta_u\Omega \equiv \mathbf{t}\,.\,\mathbf{l}_1(U) - \lambda W_1(U) = 0 \tag{6}$$

$$\delta_v\Omega \equiv \mathbf{t}\,.\,\mathbf{l}_{11}(vV) + \mathbf{m}\,.\,\mathbf{k}(V) = 0 \tag{7}$$

in which U, V cover the range‡ of kinematically admissible displacements satisfying the second and third boundary conditions in equation (3). We note that equations (6) and (7), in addition to equilibrium, imply also the natural boundary conditions. In equation (7)

† Whenever no misunderstandings can arise, the same symbols may denote densities or global values. With the exception of equation (9), inner products represent global quantities, i.e. integrals, summations over discrete members, etc.

‡ In general, classes of fields are designated by capital letters (as in U), specific fields by small letters (as in u).

and hereafter we define bilinear forms by means of the identity

$$\mathbf{l}_2(v+V) = \mathbf{l}_2(v)+2\mathbf{l}_{11}(vV)+\mathbf{l}_2(V) \tag{8}$$

and note that, as a consequence, $\mathbf{l}_{11}(vv) = \mathbf{l}_2(v)$.

Assuming equation $(4)_1$ to have the inverse $\mathbf{e} = \mathbf{e}(\mathbf{t})$ we define the complementary energy density $U_2^*(\mathbf{t})$ in the usual fashion by means of

$$U_2[\mathbf{e}(\mathbf{t})] = \mathbf{t}\,.\,\mathbf{e}(\mathbf{t})-U_2^*[\mathbf{t}] \tag{9}$$

and hence

$$\mathbf{e} = \frac{\mathrm{d}U_2^*}{\mathrm{d}\mathbf{t}}. \tag{10}$$

If $\mathbf{T}'$ comprises the class of all self-equilibrated stress systems satisfying homogeneous natural boundary conditions ($\lambda = 0$), then

$$\mathbf{l}_1\,.\,\mathbf{T}' = 0 \tag{11}$$

by equation (6); with equations $(2)_1$ and (10) this becomes the condition of compatibility

$$\left[\frac{\mathrm{d}U_2^*}{\mathrm{d}\mathbf{t}}-\frac{1}{2}\mathbf{l}_2(v)\right].\,\mathbf{T}' \equiv 2U_{11}^*(\mathbf{t}\mathbf{T}')-\frac{1}{2}\mathbf{l}_2(v)\,.\,\mathbf{T}' = 0. \tag{12}$$

Equation (7) and (12) form the basic system of equations governing the behavior of the structure in terms of the (mixed) variables v and $\mathbf{t}$.

The unbuckled state is characterized by $v = 0$ and $\mathbf{t} = \lambda\mathbf{t}_0$; equation (12) reduces to the conventional "virtual work" equation

$$U_{11}^*(\mathbf{t}_0\mathbf{T}') = 0 \tag{13}$$

while equation (7) is automatically satisfied. In the buckled state (assuming the same load parameter) the stress field is

$$\mathbf{t} = \lambda\mathbf{t}_0+\mathbf{t}' \tag{14}$$

in which $\mathbf{t}' \in \mathbf{T}'$ is self-equilibrated. In view of equations (13) and (14), equation (12) becomes

$$2U_{11}^*(\mathbf{t}'\mathbf{T}')-\tfrac{1}{2}\mathbf{l}_2(v)\,.\,\mathbf{T}' = 0 \tag{15}$$

which, with the special substitution $\mathbf{T}' = \mathbf{t}'$, implies

$$2U_2^*(\mathbf{t}') = \tfrac{1}{2}\mathbf{t}'\,.\,\mathbf{l}_2(v). \tag{16}$$

We rewrite equation (7) in the form

$$\frac{\mathrm{d}U_2'}{\mathrm{d}\mathbf{k}}\,.\,\mathbf{k}(V)+(\lambda\mathbf{t}_0+\mathbf{t}')\,.\,\mathbf{l}_{11}(vV) = 0 \tag{17}$$

and note that, for $V = v$, equation (17) implies

$$I_2(\lambda\,;\mathbf{t}'\,;\mathbf{v}) \equiv U_2'+\tfrac{1}{2}(\lambda\mathbf{t}_0+\mathbf{t}')\,.\,\mathbf{l}_2(v) = 0. \tag{18}$$

Moreover, with I_2 defined as in equation (18), equation (17) represents $\delta_v I_2 = 0$ for fixed λ and $\mathbf{t}'$ and hence constitutes a necessary, though not sufficient, condition that the actual displacement vector v minimize I_2, that is,

$$I_2(\lambda;\mathbf{t}';V) \geq I_2(\lambda;\mathbf{t}';v) = 0. \tag{19}$$

The second variation of I_2 is given by

$$\delta_v^2 I_2(\lambda;\mathbf{t}';v) = I_2(\lambda;\mathbf{t}';\delta v) = U_2'[k(\delta v)] + \tfrac{1}{2}(\lambda \mathbf{t}_0 + \mathbf{t}') \,.\, \mathbf{l}_2(\delta v). \tag{20}$$

Moreover, for given load parameter λ, the change $\Delta\Omega$ in the potential energy between the actual state and any other admissible state is

$$\Delta\Omega = I_2(\lambda;\mathbf{t}';\delta v) + U_2^*(\delta \mathbf{t}') \tag{21}$$

with δv, $\delta \mathbf{t}'$ representing the change in the associated displacement and stress fields, respectively. The last term on the right side of equation (21) is positive definite; the actual state therefore corresponds to an absolute minimum of the potential energy Ω if inequality (19) is satisfied.† Unless otherwise noted this will be postulated in what follows.‡

3. INITIAL BUCKLING ANALYSIS

With the introduction, as in [2], of the expansion parameter ε by means of

$$\begin{aligned} \delta v &= \varepsilon v_1 + \varepsilon^2 v_2 + \dots \\ \delta \mathbf{t}' &= \varepsilon \mathbf{t}_1' + \varepsilon^2 \mathbf{t}_2' + \dots \end{aligned} \tag{22}$$

equation (21) becomes

$$\begin{aligned} \Delta\Omega &= \varepsilon^2\Omega_2 + \varepsilon^3\Omega_3 + \varepsilon^4\Omega_4 + \dots \\ \Omega_2 &= I_2(\lambda;\mathbf{t}';v_1) + U_2^*(\mathbf{t}_1') \\ \Omega_3 &= 2I_{11}(\lambda;\mathbf{t}';v_1 v_2) + 2U_{11}^*(\mathbf{t}_1'\mathbf{t}_2') \\ \Omega_4 &= 2I_{11}(\lambda;\mathbf{t}';v_1 v_3) + I_2(\lambda;\mathbf{t}';v_2) + 2U_{11}^*(\mathbf{t}_1'\mathbf{t}_3') + U_2^*(\mathbf{t}_2'). \end{aligned} \tag{23}$$

Similarly, with the substitution of $\mathbf{t}'+\delta\mathbf{t}'$ for $\mathbf{t}'$ and of $v+\delta v$ for v in equation (15) and with the use of equation (22), the expanded compatibility equations

$$\begin{aligned} 2U_{11}^*(\mathbf{t}_1'\mathbf{T}') &= 0 \\ 2U_{11}^*(\mathbf{t}_2'\mathbf{T}') &= \tfrac{1}{2}\mathbf{l}_2(v_1) \,.\, \mathbf{T}' \\ 2U_{11}^*(\mathbf{t}_3'\mathbf{T}') &= \mathbf{l}_{11}(v_1 v_2) \,.\, \mathbf{T}' \end{aligned} \tag{24}$$

furnish the connection between the additional stress fields $\mathbf{t}_i'$ and the additional displacement fields v_i appearing in equations (23). We note in particular that, with $\mathbf{T}' = \mathbf{t}_1'$ and in view of the positive definiteness of U_2^*, equation $(24)_1$ implies

$$\mathbf{t}_1' = 0. \tag{25}$$

† This does not establish uniqueness, of course. For example, it follows from the complete symmetry of the structure that $\Delta\Omega = 0$ if $\delta v = -2v$, $\delta \mathbf{t}' = 0$.

‡ Exceptions (implying "secondary buckling") may occur near multiple roots of equation (17).

The unbuckled state ($\mathbf{t}' = v = 0$) becomes "critical" and incipient buckling becomes possible ($\lambda = \lambda_0$) when the smallest admissible value of Ω_2 vanishes, or, by equations $(23)_2$ and (25)

$$I_{11}(\lambda_0; 0; v_1 V) = 0; \tag{26}$$

this represents the familiar eigenvalue problem of linear buckling theory. The remaining terms in equations (23) are evaluated through the use of equations (25) and (26) (with $V = v_2$ and $V = v_3$, successively), namely,

$$\begin{aligned} \Omega_3 &= 0 \\ \Omega_4 &= I_2(\lambda_0; 0; v_2) + U_2^*(\mathbf{t}'_2) \end{aligned} \tag{27}$$

in which $\mathbf{t}'_2$ is connected to v_1 through equation $(24)_2$. Since both U_2^* and I_2 are non-negative† we conclude that *completely symmetric structures exhibit stable buckling points.*

According to Koiter [2], it follows as a corollary that this type of structure buckles under increasing (or at least not decreasing) load. This can be verified in the present case by expanding λ near the point of bifurcation,

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots \tag{28}$$

and by expanding the actual buckling deformation v and additional stress t' as in equation (22). The compatibility conditions then are identical with equations (24), while equations (17) become

$$\begin{aligned} &\mathbf{m}(v_1) \,.\, \mathbf{k}(V) + \lambda_0 \mathbf{t}_0 \,.\, \mathbf{l}_{11}(v_1 V) = 0 \\ &\mathbf{m}(v_2) \,.\, \mathbf{k}(V) + \lambda_0 \mathbf{t}_0 \,.\, \mathbf{l}_{11}(v_2 V) + (\lambda_1 \mathbf{t}_0 + \mathbf{t}'_1) \,.\, \mathbf{l}_{11}(v_1 V) = 0 \\ &\mathbf{m}(v_3) \,.\, \mathbf{k}(V) + \lambda_0 \mathbf{t}_0 \,.\, \mathbf{l}_{11}(v_3 V) + (\lambda_1 \mathbf{t}_0 + \mathbf{t}'_1) \,.\, \mathbf{l}_{11}(v_2 V) + (\lambda_2 \mathbf{t}_0 + \mathbf{t}'_2) \,.\, \mathbf{l}_{11}(v_1 V) = 0. \end{aligned} \tag{29}$$

From equation $(24)_1$ (with $\mathbf{T}' = \mathbf{t}'_1$) it follows that $\mathbf{t}'_1 = 0$ as before. Identifying ε through

$$\begin{aligned} &\tfrac{1}{2}\mathbf{t}_0 \,.\, \mathbf{l}_2(v_1) = -1 \\ &\mathbf{t}_0 \,.\, \mathbf{l}_{11}(v_1 v_i) = 0 \qquad (i = 2, 3, \dots) \end{aligned} \tag{30}$$

and subtracting equation $(29)_1$ (with $V = v_2$) from equation $(29)_2$ (with $V = v_1$) we establish

$$\lambda_1 = 0 \tag{31}$$

as expected. Applying the same procedure to equations $(29)_1$ and $(29)_3$ (with $V = v_3$ and $V = v_1$, respectively) and making use of equations (30) and (31) results in

$$2\lambda_2 = \mathbf{t}'_2 \,.\, \mathbf{l}_2(v_1). \tag{32}$$

In view of equation $(24)_2$ (with $\mathbf{T}' = \mathbf{t}'_2$) this implies‡

$$\lambda_2 = 2U_2^*(\mathbf{t}'_2) \geq 0. \tag{33}$$

† The latter since $I_2(\lambda_0; 0; v_1) = 0$ is the smallest possible value.

‡ Apparent counterexamples have been cited by Britvec and Chilver [3], among others. We note, however, that if the order of magnitude assumptions made in the present paper are applied to the statically determinate trusses analyzed by these authors, then their (relatively minor) initial instability phenomenon disappears and the trusses buckle under constant load. In contrast, statically indeterminate trusses may buckle under substantially increasing loads, as pointed out by Masur [12].

It can be shown that the load parameter ($\lambda > 0$) is monotonically increasing (or at least non-decreasing) throughout the postbuckling history. That is, if the "path length" S is defined by

$$S \equiv -\tfrac{1}{2}\mathbf{t}_0 \,.\, \mathbf{l}_2(v), \tag{34}$$

then

$$\frac{d\lambda}{dS} \geq 0 \tag{35}$$

for any continuous process.†

4. POST-BUCKLING ANALYSIS, GEOMETRIC INTERPRETATION, AND BOUNDEDNESS THEOREMS

In the analysis of the post-buckling behavior it is convenient and instructive to introduce geometric analogies which appeal to visual intuition without losing rigor. We may, for example, construct a function space which is based on the stress field $\mathbf{t}$ in such a way that if $\mathbf{t}_1$ and $\mathbf{t}_2$ are any two stress fields associated with the strain fields $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively, through equation (10), then the inner product $(\mathbf{t}_1, \mathbf{t}_2)$ is defined by means of

$$(\mathbf{t}_1, \mathbf{t}_2) \equiv \mathbf{t}_1 \,.\, \mathbf{e}_2 = 2U^*_{11}(\mathbf{t}_1\mathbf{t}_2) = \mathbf{t}_2 \,.\, \mathbf{e}_1 = (\mathbf{t}_2, \mathbf{t}_1). \tag{36}$$

Equation (36), with $\mathbf{t}_1 = \mathbf{t}_2$, defines the "distance" in the function space; moreover, because of the Schwarz inequality, the angle between two vectors is real.

In particular let $\mathbf{T}'$ represent the class of all "self-equilibrated" stress fields satisfying

$$\mathbf{T}' \,.\, \mathbf{l}_1(U) = 0 \tag{37}$$

for all displacement fields U, and satisfying, in addition, the homogeneous boundary conditions on B_t. Let $\mathbf{T}''$ represent the class of all "self-compatible" stress fields satisfying

$$\frac{dU^*_2}{d\mathbf{t}}(\mathbf{T}'') = \mathbf{l}_1(u'') \qquad u'' = 0 \text{ on } B_u. \tag{38}$$

It then follows that the two classes of stress fields are mutually orthogonal in the sense that

$$(\mathbf{T}', \mathbf{T}'') = 0 \tag{39}$$

for any pair of members.

They are also "complete" in the sense that any given stress field $\mathbf{T}$ may be split uniquely into

$$\mathbf{T} = \mathbf{T}' + \mathbf{T}''. \tag{40}$$

In fact, it follows from equations (40) and (37) that

$$\mathbf{T}'' \,.\, \mathbf{l}_1(U) = \mathbf{T} \,.\, \mathbf{l}_1(U) \tag{41}$$

† This is proved through a procedure which is similar to the one employed in the establishment of equations (31) and (33). We start again with the basic equations (15) and (17), which we differentiate with respect to S, and make use of the inequality (19).

for all kinematically admissible displacement fields U. This represents the equations of equilibrium and the natural boundary conditions on B_t, with the right side taking the place of a body force field or a boundary traction field, respectively. Since equations (38) constitute the stress–displacement equations and boundary conditions on B_u the system of equations governing $\mathbf{T}''$ and u'' is presumed to be determinate. Equation (40) finally serves to find $\mathbf{T}'$.

We now consider the space of all self-equilibrated stress fields $\mathbf{T}'$. For given load parameter λ a "stable" point Q may be identified by the inequality

$$I_2(\lambda\,;\mathbf{T}'_Q\,;V) > 0 \tag{42}$$

for all kinematically admissible displacements V which do not vanish trivially. If there exists at least one displacement field V for which I_2 becomes negative, a point is defined to be "unstable". The totality of all "critical" points P satisfying

$$I_2(\lambda;\mathbf{T}'_P;V) \geq I_2(\lambda;\mathbf{T}'_P;V_P) = 0 \tag{43}$$

then forms a "hypersurface" S which separates the stable and unstable regions. We note that, by equation (19), the actual stress $\mathbf{t}'$ lies on S, with $V_P \equiv v$.

It is easy to show that S is convex. For given λ let Q be a stable point, and let P be a critical point satisfying condition (43). By inserting $V = V_P$ into inequality (42) and by subtracting equation (43) it follows that

$$I_2(0;\mathbf{T}'_Q-\mathbf{T}'_P;V_P) = \tfrac{1}{2}(\mathbf{T}'_Q-\mathbf{T}'_P)\,.\,\mathbf{l}_2(V_P) > 0. \tag{44}$$

If now a point R is identified by

$$\mathbf{T}'_R = \mathbf{T}'_P+\alpha(\mathbf{T}'_P-\mathbf{T}'_Q) \qquad (\alpha > 0)$$

then

$$I_2(\lambda;\mathbf{T}'_R;V_P) < 0.$$

R is therefore an unstable point, and the line QPR cuts S only at P. Since both Q and P are arbitrary the convexity of S has been proved.

If V_P in equation (43) is unique (except for an arbitrary multiplier) we define the strain field $\mathbf{e}_P$ and the associated stress field $\mathbf{N}_P$ (and its self-equilibrated and self-compatible "components") by means of

$$\begin{aligned} \mathbf{e}_P &= \tfrac{1}{2}\mathbf{l}_2(V_P) \\ \mathbf{N}_P \equiv \mathbf{t}(\mathbf{e}_P) &= \mathbf{N}'_P+\mathbf{N}''_P \\ (\mathbf{N}'_P,\mathbf{N}'_P) &= 1 \end{aligned} \tag{45}$$

in which the normalizing condition has been added for the sake of definiteness. From equations (44), (45) and (39) it follows that

$$(\mathbf{T}'_Q-\mathbf{T}'_P,\mathbf{N}'_P) > 0 \tag{46}$$

for all stable points Q; this implies† that $\mathbf{N}'_P$ is normal to S at P and points "inward".

If V_P is not unique, say, V_1 or V_2, then, in general,

$$\begin{aligned} V_P &= \alpha_1 V_1 + \alpha_2 V_2 \\ \mathbf{l}_2(V_P) &= \alpha_1^2 \mathbf{l}_2(V_1) + 2\alpha_1\alpha_2 \mathbf{l}_{11}(V_1 V_2) + \alpha_2^2 \mathbf{l}_2(V_2) \\ \mathbf{N}'_P &= \alpha_1^2 \mathbf{N}'_{11} + 2\alpha_1\alpha_2 \mathbf{N}'_{12} + \alpha_2^2 \mathbf{N}_{22} \end{aligned} \tag{47}$$

and $\mathbf{N}'_P$ sweeps out an elliptic conical surface (which flattens to a plane if $\mathbf{N}'_{11}$, $\mathbf{N}'_{12}$, $\mathbf{N}'_{22}$ exhibit linear dependence). Equation (46) then implies that there is a manifold of normals to S which lie inside or on the elliptic cone (or, in the special case, on the plane). Conversely, S exhibits an apex whose tangents form a cone which is perpendicular to the cone formed by the normals.

For the sake of definiteness‡ let

$$\mathbf{t}_0 \,.\, \mathbf{l}_2(V) < 0 \tag{48}$$

for all V, and let P be a critical point associated with the loading parameter λ. For $\lambda' > \lambda$, it follows from equations (43) and (48) that $I_2(\lambda'; \mathbf{T}'_P; V_P) < 0$ and hence that P lies "outside" the critical surface S identified with λ', or, equivalently, that critical surfaces S lie "inside" one another for increasing values of λ.

We now insert $\mathbf{T}' = \mathbf{T}'_Q - \mathbf{t}'$ in equation (15), with Q representing any stable or critical point and $\mathbf{t}'$ the actual stress, and consider equation (44) (with $\mathbf{T}'_P = \mathbf{t}'$, $V_P = v$). Then

$$(\mathbf{t}', \mathbf{t}') \leq (\mathbf{t}', \mathbf{T}'_Q) \tag{49}$$

or, with $\mathbf{t}' \neq \mathbf{T}'_Q$,

$$(\mathbf{t}', \mathbf{t}') < (\mathbf{T}'_Q, \mathbf{T}'_Q). \tag{50}$$

In other words, the actual stress point is closer to the origin than any other point either on or inside S. This represents an extension of the familiar Castigliano principle of minimum complementary energy to the non-linear post-buckling case.

Figure 1 represents the post-buckling history in the space of the $\mathbf{T}'$ stresses. For $\lambda < \lambda_0$ the origin itself is stable and therefore coincides with the actual stress point in the unbuckled state, as expected. For $\lambda > \lambda_0$ the actual stress point "travels" with the associated critical surface S in such a way as always to be as close as possible to 0. We note that the stress path C is not necessarily normal to S except for incipient buckling when $\lambda = \lambda_0$; in that case v approaches v_1 and $\mathbf{t}'$ points in the direction of $\mathbf{N}'$.

In general the analysis of the buckled state ($\lambda > \lambda_0$) presents formidable analytical difficulties. By means of an approximate procedure, however, it is possible to bracket the magnitude of $\mathbf{t}'$ between lower and upper bounds which may be narrowed to within any prescribed limit. An error estimate is therefore available as well as a scheme of reducing the error.

Let $\mathbf{T}'_u$ represent any self-equilibrated normalized stress field [i.e. $(\mathbf{T}'_u, \mathbf{T}'_u) = 1$] and assume that $\mathbf{T}'_P = \beta\mathbf{T}'_u(\beta > 0)$ defines a point P on S by satisfying the relationship (43).

† Actually the convexity of S can also be deduced directly.

‡ This may require restrictions on the class of admissible displacement fields, as for example, in the case of plates subjected to shearing stresses, beams buckling laterally, etc. Unless such restrictions are imposed the spectrum of eigenvalues may include negative values.

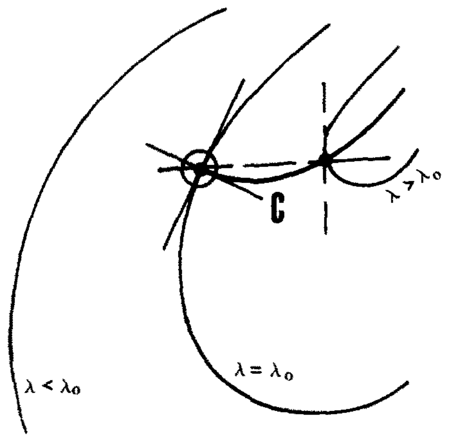

FIG. 1. Space of self-equilibrated stresses.

With $\mathbf{N}'_P$ defined as in equation (45) the magnitude of the actual stress field $\mathbf{t}'$ is then bounded from both below and above by the inequalities

$$\tfrac{1}{2}\mathbf{T}'_P \cdot \mathbf{l}_2(V_P) \equiv (\mathbf{T}'_P, \mathbf{N}'_P) \leq_+ (\mathbf{t}', \mathbf{t}')^{\frac{1}{2}} \leq \beta. \tag{51}$$

The proof of these bounds is facilitated by consulting Fig. 2. The second inequality follows directly from the convexity of S or, equivalently, from the inequality (50). The term on the left side of the inequality (51) represents the distance $0P''$. This is the shortest distance from 0 to any point on the tangent plane; in view again of the convexity of S this proves the first inequality.

The calculation of β and V_P by means of equation (43) constitutes a linear eigenvalue problem which may still involve substantial numerical labor. This can be reduced further through the application of a "modified Rayleigh" process based on a corollary to the first of the inequalities (51). Assume any normalized stress field $\mathbf{T}'_u$ and any displacement field V_R subject to the normalizing condition $(\mathbf{N}'_R, \mathbf{N}'_R) = 1$ and subject also to the restriction

$$\tfrac{1}{2}\mathbf{T}'_u \cdot \mathbf{l}_2(V_R) = (\mathbf{T}'_u, \mathbf{N}'_R) > 0.$$

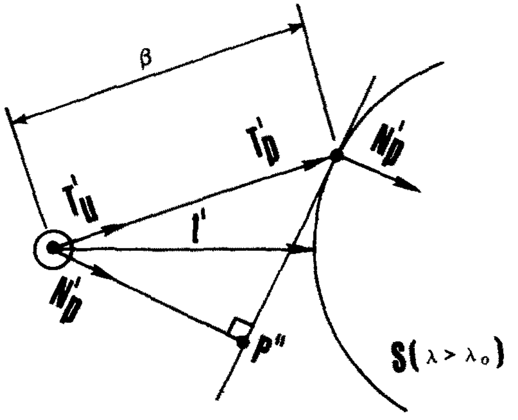

FIG. 2. Upper and lower bounds in stress space.

If $\mathbf{T}'_R = \beta_R \mathbf{T}'_u$, with the "Rayleigh coefficient" $\beta_R > 0$ satisfying

$$I_2(\lambda;\mathbf{T}'_R;V_R) = 0, \tag{52}$$

then

$$\tfrac{1}{2}\mathbf{T}'_R\,.\,\mathbf{l}_2(V_R) = (\mathbf{T}'_R,\mathbf{N}'_R) \leq_+ (\mathbf{t}',\mathbf{t}')^{\frac{1}{2}}. \tag{53}$$

This establishes a simple lower bound to the magnitude of the actual stress field $\mathbf{t}'$.

The proof follows from the fact that by equation (52) the stress point R lies either on or outside of S. Moreover, all points $\mathbf{T}'$ satisfying

$$I_2(\lambda;\mathbf{T}';V_R) = 0 \tag{54}$$

lie on a hyperplane which is normal to the vector $\mathbf{N}'_R$ and which passes through R since, by equations (52) and (54),

$$I_2(0;\mathbf{T}'-\mathbf{T}'_R;V_R) \equiv (\mathbf{T}'-\mathbf{T}'_R,\mathbf{N}'_R) = 0.$$

Since the hyperplane lies everywhere outside of S, inequality (53) represents a corollary of the first inequality (51). We note, however, that no upper bound has been established since $\beta_R \leq \beta$.

If the unit vectors $\mathbf{T}'_u$ and $\mathbf{N}'_P$ satisfy $(\mathbf{T}'_u\,.\,\mathbf{N}'_P) = 1$ and are identical, then $\mathbf{T}'_P$ represents the correct stress field $\mathbf{t}'$, and the inequalities (51) become identities. We also note that if the eigenvalue problem equation (43) has a multiple root [as, for example, in equation (47)], then the largest lower bound in the inequality (51) is obtained by maximizing $(\mathbf{T}'_P,\mathbf{N}'_P)$.

5. INITIAL IMPERFECTIONS

In Section 3 it was shown that completely symmetric structures exhibit stable points of bifurcation. Unlike the case of "imperfection-sensitive" structures (such as shells or rings under special conditions), the effect of initial imperfections on the buckling strength is therefore generally moderate and never catastrophic. Nevertheless, a study of initial imperfections is instructive and included for completeness.

In the spirit of the order-of-magnitude assumptions made previously it is reasonable to postulate only initial imperfections v^* (and not u^*). While equation (1) remains valid, equations (2) are replaced by

$$\begin{aligned} \mathbf{e} &= \mathbf{l}_1(u)+\mathbf{l}_{11}(v^*v)+\tfrac{1}{2}\mathbf{l}_2(v) \\ \mathbf{k} &= \mathbf{k}(v). \end{aligned} \tag{55}$$

The constitutive equations (4) and (10) are also unchanged, as is equation (14), in which $\mathbf{t}_0$ continues to be interpreted as the stress in the unbuckled perfect structure subjected to a unit load parameter. The actual behavior is governed by the compatibility and equilibrium conditions, respectively,

$$2U^*_{11}(\mathbf{t}'\mathbf{T}')-[\mathbf{l}_{11}(v^*v)+\tfrac{1}{2}\mathbf{l}_2(v)]\,.\,\mathbf{T}' = 0 \tag{56}$$

$$\mathbf{m}(v)\,.\,\mathbf{k}(V)+(\lambda\mathbf{t}_0+\mathbf{t}')\,.\,\mathbf{l}_{11}(vV) = -(\lambda\mathbf{t}_0+\mathbf{t}')\,.\,\mathbf{l}_{11}(v^*V). \tag{57}$$

We distinguish several possibilities depending on the order-of-magnitude of the initial imperfection v^* and of the buckling displacement v.

Case 1. Small imperfections and displacements

We replace v^* by εv^* and v by εv, and linearize the equations relative to the small parameter ε. By equation (56) this implies that $\mathbf{t}'$ is replaced by $\varepsilon^2\mathbf{t}'$, and equation (57), after linearization, becomes

$$\mathbf{m}(v) \,.\, \mathbf{k}(V) + \lambda \mathbf{t}_0 \,.\, \mathbf{l}_{11}(vV) = -\lambda \mathbf{t}_0 \,.\, \mathbf{l}_{11}(v^*V). \tag{58}$$

Since equation (26) can be written in the form

$$\mathbf{m}(v_1) \,.\, \mathbf{k}(V) + \lambda_0 \mathbf{t}_0 \,.\, \mathbf{l}_{11}(v_1 V) = 0 \tag{59}$$

a comparison shows that if the initial imperfection v^* is affine to the buckling mode v_1 of the perfect structure then the displacement of the actual structure is also affine to v_1, that is,

$$v = \frac{1}{\left(\dfrac{\lambda_0}{\lambda}\right) - 1} v^*. \tag{60}$$

This is the basis for the familiar Southwell plot which is applicable to linear theory.

Case 2. Small imperfections, finite displacements

If v^* is replaced by εv^*, but v is held finite, then equations (56) and (57) degenerate to the equations governing the perfect structure. This confirms the well-known fact that the extended post-buckling behavior of structures is not affected by small initial imperfections.

Case 3. Finite initial imperfections, small displacements

In that case the lowest order terms for v, $\mathbf{t}'$ and λ start with ε, and equations (56) and (57) become, respectively,

$$\begin{aligned} 2U^*_{11}(\mathbf{t}'\mathbf{T}') &= \mathbf{T}' \,.\, \mathbf{l}_{11}(v^*v) \\ \mathbf{m}(v) \,.\, \mathbf{k}(V) &= -(\lambda \mathbf{t}_0 + \mathbf{t}') \,.\, \mathbf{l}_{11}(v^*V). \end{aligned} \tag{61}$$

This system of equations is typical of the linearized equations of shallow shell theory, with v^* taking the place of the deviation from flatness.

Case 4. Special case

Let

$$\mathbf{t}_0 \,.\, \mathbf{l}_{11}(v^*V) = 0 \tag{62}$$

as, for example, in the case of a shallow cylindrical shell segment under axial compression or torsion; then equations (56) and (57) admit the "unbuckled" solution

$$v = \mathbf{t}' = 0 \tag{63}$$

as if the structure were initially perfect. If now δv and $\delta \mathbf{t}'$ are expanded as in equation (22), then $\Delta\Omega$ in equation (23) is unchanged, whereas compatibility conditions equations (24) become

$$(\mathbf{t}'_1, \mathbf{T}') = \mathbf{l}_{11}(v^*v_1) \,.\, \mathbf{T}'$$
$$(\mathbf{t}'_2, \mathbf{T}') = [\mathbf{l}_{11}(v^*v_2) + \tfrac{1}{2}\mathbf{l}_2(v_1)] \,.\, \mathbf{T}' \tag{64}$$
$$(\mathbf{t}'_3, \mathbf{T}') = [\mathbf{l}_{11}(v^*v_3) + \mathbf{l}_{11}(v_1 v_2)] \,.\, \mathbf{T}'.$$

Bifurcation occurs again when the smallest admissible value of Ω_2 vanishes; by equation $(23)_2$ this implies

$$2I_{11}(\lambda_0; 0; v_1 V) + (\mathbf{t}'_1; \mathbf{T}'_1) = 0 \tag{65}$$

for all admissible V and provided $\mathbf{T}'_1$ is associated with V through

$$(\mathbf{T}'_i, \mathbf{T}') = \mathbf{l}_{11}(v^*V) \,.\, \mathbf{T}' \qquad (i = 1, 2, \ldots) \tag{66}$$

which represents the variation of equation $(64)_1$. In particular, if $V \equiv v_1$ then, by equations (66) and $(64)_1$, $\Omega_2 = 0$ as expected; however, $U_2^*(\mathbf{t}'_1) > 0$, in general, and hence $I_2 < 0$. By equation (48) this implies that the critical buckling load λ_0 is raised as a result of the initial imperfection. Ω_3 is obtained from equation $(23)_3$. By equation (65), with $V = v_2$,

$$\Omega_3 = (\mathbf{t}'_1, \mathbf{t}'_2) - (\mathbf{t}'_1, \mathbf{T}'_1)$$

in which $\mathbf{T}'_1$ is compatible with v_2. Using equation (66) (with $V = v_2, \mathbf{T}' = \mathbf{t}_1$) we find

$$\Omega_3 = \tfrac{1}{2}\mathbf{t}'_1 \,.\, \mathbf{l}_2(v_1). \tag{67}$$

Similarly, from equation $(23)_4$ and the use of (equations (65), (66) ($V = v_3$), ($T' = t_1$) and $(64)_3$ ($T' = t_1$),

$$\Omega_4 = I_2(\lambda_0; 0; v_2) + U_2^*(\mathbf{t}'_2) + \mathbf{t}'_1 \,.\, l_{11}(v_1 v_2). \tag{68}$$

If $\Omega_3 \neq 0$ the point of bifurcation is necessarily unstable. For $\Omega_3 = 0$ the symmetry (although not the "complete symmetry") of the structure is preserved, and stability is determined by minimizing Ω_4. This leads to the deformation mode v_2 governed by

$$2I_{11}(\lambda; 0 : v_2 V) + (\mathbf{t}'_2, \mathbf{T}'_2) + \mathbf{t}'_1 \,.\, \mathbf{l}_{11}(v_1 V) = 0 \tag{69}$$

in which

$$(\mathbf{t}'_2, \mathbf{T}'_2) = \mathbf{l}_{11}(v^*V) \,.\, \mathbf{t}'_2 \tag{70}$$

(obtained from equation $(66)_2$ by inserting $\mathbf{t}'_2$ for $\mathbf{T}'$) represents the connection between $\mathbf{T}'_2$ and V. Eventually,

$$(\Omega_4)_{\min} = \tfrac{1}{2}\mathbf{t}'_1 \,.\, \mathbf{l}_{11}(v_1 v_2) + \tfrac{1}{4}\mathbf{t}'_2 \,.\, \mathbf{l}_2(v_1), \tag{71}$$

and the critical point is stable or unstable according to the sign of $(\Omega_4)_{\min}$. An extensive discussion of this question was first presented by Koiter in [2].

6. LIMIT ANALYSIS

Under certain circumstances the hyper-surfaces S may be closed; in that case they may shrink to a point as the load parameter and associated stress field approach the finite values of $\lambda = \lambda_c$ and $\mathbf{t}' = \mathbf{t}'_c$, respectively, while the displacements increase indefinitely.

Let this collapse mechanism be identified by

$$v = Mv_c$$
$$\tfrac{1}{2}\mathbf{t}_0 \,.\, \mathbf{l}_2(v_c) = -1 \tag{72}$$

with $M \to \infty$ and equation (15) becoming

$$\mathbf{l}_2(v_c) \,.\, \mathbf{T}' = 0 \tag{73}$$

for all self-equilibrated stress fields $\mathbf{T}'$. [Equivalently this means that the stress field associated with $\mathbf{l}_2(v_c)$ is self-compatible.] Equation (17) becomes

$$\mathbf{m}(v_c) \,.\, \mathbf{k}(V) + (\lambda_c \mathbf{t}_0 + \mathbf{t}_c') \,.\, \mathbf{l}_{11}(v_c V) = 0 \tag{74}$$

which, with $V = v_c$ and equation (73) implies

$$I_2(\lambda_c\,; 0\,; v_c) = 0\,. \tag{75}$$

Because of the nonlinearity of the governing equations the determination of the collapse load λ_c and associated collapse mechanism v_c and stress field $\mathbf{t}_c'$ may present formidable analytical difficulties. These can be partially circumvented by "bracketing" the actual collapse parameter λ_c between a class of "kinematically admissible" load parameters λ_k and a class of "statically admissible" load parameters λ_s, of which the former represent upper bounds and the latter lower bounds to the actual collapse parameter.†

Define a kinematically admissible collapse mechanism v_k and associated kinematically admissible load parameter λ_k by

$$\mathbf{l}_2(v_k) \,.\, \mathbf{T}' = 0 \qquad \text{(for all } \mathbf{T}'\text{)} \tag{76}$$

$$I_2(\lambda_k\,; 0\,; v_k) \le 0, \tag{77}$$

in which equation (77) implies that, during collapse, the external rate of work at least equals the internal energy dissipation.

Define a load parameter λ_s to be "statically admissible" if there exists at least one stress field $\mathbf{t}_s'$ such that

$$I_2(\lambda_s\,; \mathbf{t}_s'\,; V) \ge 0 \qquad \text{(for all } V\text{)}. \tag{78}$$

This implies that, with the stress field so defined, the structure is stable. We note that neither the displacement field v_k nor the stress field $\mathbf{t}_s'$ need be the actual fields at collapse.

If equation (78) (with $V = v_k$) is subtracted from equation (77) and equation (76) is considered, then

$$I_2(\lambda_s - \lambda_k\,; 0\,; v_k) \ge 0. \tag{79}$$

Since $U_2' > 0$ this implies that $\lambda_k \ge \lambda_s$. Moreover, since the actual collapse parameter λ_c is both kinematically and statically admissible,

$$\lambda_k \ge \lambda_c \ge \lambda_s\,. \tag{80}$$

† The language used is obviously borrowed from perfect plasticity theory. This is not accidental since, in essence, all proofs introduced in the present study are based on the convexity of the critical surfaces S, while the limit theorems of perfect plasticity are based on the convexity of the yield surface. The difference between the two approaches is only one of selecting the proper space.

THEOREM.

The collapse load parameter is the smallest of all kinematically admissible and the largest of all statically admissible load parameters.

A necessary condition for the existence of a collapse load has not been established as yet. We note, however, that the existence of a kinematically admissible load parameter and collapse mechanism satisfying equations (76) and (77) is sufficient.

7. EXAMPLE

As a demonstration example we introduce the case of a narrow rectangular beam of width b and depth h which is subjected to a bending moment M in its plane of major stiffness (see Fig. 3). The beam is fixed in its major plane at the far end and is therefore statically indeterminate to the first degree relative to its major bending moments. To simplify the calculations, however, it is assumed to be simply supported at both ends with regard to lateral deflections and rotations.

In conformity with the notation selected in the remainder of the present study we let e_1 and e_2 represent the axial strain and the curvature in the major plane, respectively, and t_1 and t_2 the associated axial force and major bending moment. Also, k_1 and k_2 represent, respectively, the curvature in the minor plane and the twist, and the associated minor bending moment and torque are identified by m_1 and m_2, respectively. With this notation

$$\begin{aligned} U_2^* &= \frac{1}{2}\int_0^L \left(\frac{t_1^2}{A_1}+\frac{t_2^2}{A_2}\right) \mathrm{d}x \\ U_2' &= \frac{1}{2}\int_0^L (B_1k_1^2+B_2k_2^2)\,\mathrm{d}x \end{aligned} \tag{81}$$

and

$$\begin{aligned} e_1 &= \frac{t_1}{A_1} \qquad & e_2 &= \frac{t_2}{A_2} \\ m_1 &= B_1k_1 \qquad & m_2 &= B_2k_2. \end{aligned} \tag{82}$$

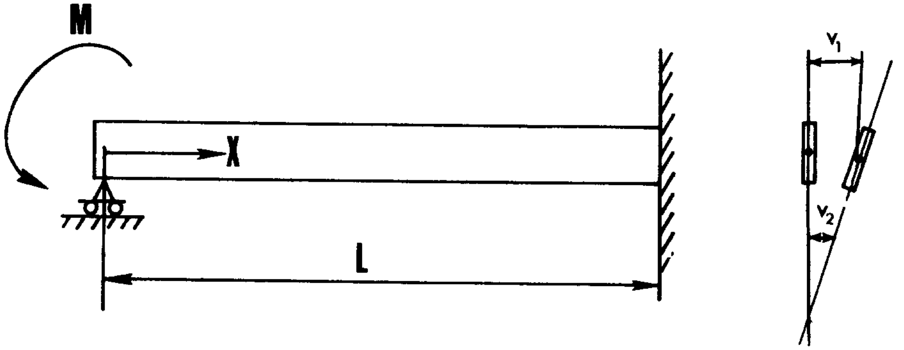

FIG. 3. Illustrative example: lateral buckling of rectangular beam.

The strain–displacement relations are

$$\begin{aligned} e_1 &= u_1' + \tfrac{1}{2}(v_1'^2 + g^2 v_2'^2) \\ e_2 &= u_2'' - v_1'' v_2 \\ k_1 &= v_1'' \\ k_2 &= v_2' \end{aligned} \tag{83}$$

in which (again in conformity with the notation employed throughout this paper) u_1 and u_2 are the axial and major bending displacement, respectively, v_1 represents the bending displacement in the minor direction, v_2 the rotation, primes (as in u_1') derivatives with respect to x, and $g = (A_2/A_1)^{\frac{1}{2}}$ is the radius of gyration.

Because of the rollers at the left end we set $t_1 = 0$; this identifies u_1 in terms of v_1 and v_2. After dropping the subscript 2 the major bending moment t then takes the form

$$t(x) = M\left[1-(1+r)\frac{x}{L}\right] \tag{84}$$

with the generalized (single parameter) self-equilibrated bending moment

$$T = \frac{x}{L} \tag{85}$$

and the compatibility condition equation (12)

$$\int_0^L \frac{t}{A_2} x\,\mathrm{d}x + \int_0^L v_1'' v_2 x\,\mathrm{d}x = 0. \tag{86}$$

In the unbuckled state the second integral in equation (86) vanishes; substitution of equation (84) in equation (86) then leads to $r = \frac{1}{2}$, as expected. In the buckled state the bending moment can therefore be written in the form

$$t = M\left(1 - \frac{3}{2}\frac{x}{L}\right) + c\frac{x}{L}, \tag{87}$$

with

$$c = -\frac{3A_2}{L^2}\int_0^L v_1'' v_2 x\,\mathrm{d}x \tag{88}$$

obtained by inserting equation (87) into equation (86).

Equation (18) becomes

$$I_2 = \frac{1}{2}\int_0^L (B_1 v_1''^2 + B_2 v_2'^2 - 2t v_1'' v_2)\,\mathrm{d}x \tag{89}$$

and, through variation,

$$\begin{aligned} B_1 v_1'' - t v_2 &= 0 \\ t v_1'' + B_2 v_2'' &= 0 \end{aligned} \tag{90}$$

in which the first equation has already been integrated twice, the constants of integration vanishing on account of the boundary conditions. If v_1 is eliminated through the use of the first of equations (90) we are finally led to

$$v_2'' + \frac{t^2}{B_1 B_2} v_2 = 0 \qquad (0 \le x \le L)$$
$$v_2(0) = v_2(L) = 0 \tag{91}$$

in which t is given in equation (87) and c can be expressed in the form

$$c = -\frac{3}{L^2}\frac{h^2}{b^2}\int_0^L t v_2^2 x \, dx. \tag{92}$$

Equation (91) has been solved for various values of c by Salvadori [13]. The solutions identify the eigenvalue M and the buckling mode v_2; the amplitude of v_2 is related to c through the compatibility condition equation (92).

The results are shown in Fig. 4, in which the non-dimensionalized applied moment is shown as a function of the amplitude of v_2. As expected, the value of the coefficient of M increases during buckling and reaches a limiting value of 8·25 at collapse. Since the corresponding value at incipient buckling is 7·32, this represents an increase of about 13 per cent.

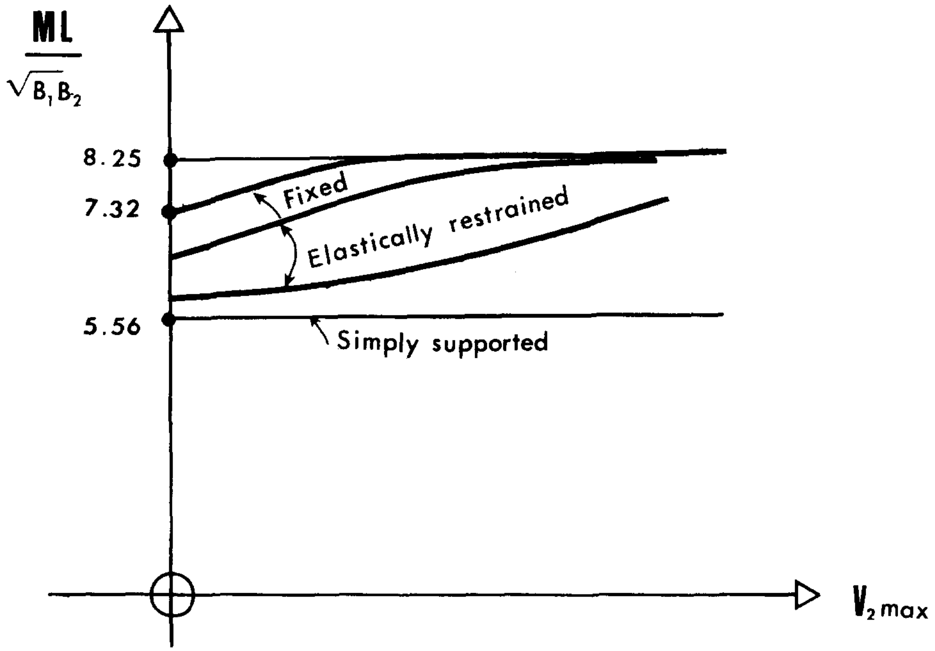

FIG. 4. Load vs. buckling amplitude.

Of more practical significance is the fact that if the right end of the beam is not fixed but elastically restrained against rotation in its major plane (by being attached, for example, to a column or to another beam), then the value of the critical moment coefficient of buckling is less than 7·32, its minimum value being 5·56 as the stiffness of the right support approaches zero. Nevertheless, for any non-vanishing support stiffness collapse occurs at the same limit value of 8·25, although the rotation required to approach this value increases with diminishing stiffness. This behavior pattern is analogous to a similar pattern which has been predicted and experimentally confirmed previously by Masur and Milbradt [14].

REFERENCES

[1] S. TIMOSHENKO, *Theory of Elastic Stability*. McGraw-Hill (1936).

[2] W. T. KOITER, On the stability of elastic equilibrium. Thesis, Delft, H. J. Paris, Amsterdam (1945). (In Dutch, English Translation: NASA TTF-10833, 1967).

[3] S. J. BRITVEC and A. H. CHILVER, Elastic buckling of rigid-jointed braced frames. *J. Engng Mech. Div., Am. Soc. Civ. Engrs* **89**, 217 (1963).

[4] J. M. T. THOMPSON, Basic principles in the general theory of elastic stability. *J. Mech. Phys. Solids* **11**, 13 (1963).

[5] J. M. T. THOMPSON and A. C. WALKER, The non-linear perturbation analysis of discrete structural systems. *Int. J. Solids Struct.* **4**, 757 (1968).

[6] M. J. SEWELL, A general theory of equilibrium paths through critical points. *Proc. R. Soc.* **A306**, 201 (1968); **A306**, 225 (1968).

[7] W. J. SUPPLE, On the change in buckle pattern in elastic structures, *Int. J. mech Sci.* **10**, 737 (1968).

[8] B. BUDINASKY and J. C. AMAZIGO, Initial post-buckling behavior of cylindrical shells under external pressure. *J. Math. Phys.* **47**, 223 (1968).

[9] B. BUDIANSKY and J. W. HUTCHINSON, Dynamic buckling of imperfection-sensitive structures. *Proc. XI int. Congr. appl. Mech.* edited by H. GORTLER. Springer (1964).

[10] W. T. KOITER, The nonlinear buckling problem of a complete spherical shell under uniform external pressure. Report No. 412, Lab. of Engng Mech., Dept. of Mech. Engng, Technical University, Delft (1968).

[11] D. BEATY, A theoretical and experimental study of the stability of thin elastic spherical shells. Thesis, Dept. of Engng Mech., University of Michigan (1962).

[12] E. F. MASUR, Post-buckling strength of redundant trusses. *Trans. Am. Soc. Civ. Engrs* **119**, 699 (1954).

[13] M. G. SALVADORI, Lateral buckling of beams of rectangular cross-section under bending and shear. *Proc. First U.S. natn. Congr. appl. Mech.* p. 403. American Society of Mechanical Engineers (1952).

[14] E. F. MASUR and K. P. MILBRADT, Collapse strength of redundant beams after lateral buckling. *J. appl. Mech.* **24**, 283 (1957).

(*Received* 26 *May* 1969)

Абстракт—Определяется, что форма порма потери устойчивости является симметрической, если ее знак неопределен. Зто происходит, когда разложение потенциальной знергии близи точки потери устойчивости не содержит кубических членов в выражении для формы потери устойчивости. Еслиже, кубипеские плены исчезают идентично для всех возможных форм потери устойчивости, тогда система является полно симметрическац. Зто определение заклюпает большинство систем, имеющих технические значение, таких как колонны, пластинки, фермы и др.

Если прилят некоторые технически реальные, в смысле порядка величиеы, предполокения, то можно привести анализ потери устойчивости для полно симметрических систем с большой общностью. На пример, в настоящей работе, приводится, что системы зтого типа начинают терять устойчивость под влиянием роста нагрузки и затем они остановливаются нечувствительными на начальные неточности. Состояние после потери устойчивости, характеризуется выполнением принципа минимума дополнительной знергии, которое является продолжением соответсвующего классического принципа на нелинейную область. Кроме того, знергия может быть заключена между верхним и нижним прелелом. Оценка погрешности основывается, таким образом, не срелнем значении.

В некоторых условиях, нагрузка приближается к конечному знауению когда система близка к разрушению. Нагрузка разрушения может быть также заклочена между параметрами нагрузки "статически допустимыми" /которые представляют нижние пределы/ и параметрами нагрузки "кинематически допустимыми" /которые представляют верхние пределы/. Можно, произвольно, сокращить илтервал между указанными пределами.

Даются пример гибкой статически неопределимой балки, подверженной действию бокого и крутильного выпучивания, чтобы показать общие принципы, предлагаемые в работе.